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Twisted classical Poincaré algebras

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Abstract. We consider the twisting of the Hopf structure for the classical enveloping algebra $U(\hat{g})$, where $\hat{g}$ is an inhomogenous rotation algebra, with explicit formulae given for the $D = 4$ Poincaré algebra ($\hat{g} = \mathcal{P}_4$). The comultiplications of twisted $U^F(\mathcal{P}_4)$ are obtained by conjugating the primitive classical coproducts by $F \in U(\hat{c}) \otimes U(\hat{c})$, where $\hat{c}$ denotes any Abelian subalgebra of $\mathcal{P}_4$, and the universal R-matrices for $U^F(\mathcal{P}_4)$ are triangular. As an example we show that the quantum deformation of the Poincaré algebra recently proposed by Chaichian and Demiczev is a twisted classical Poincaré algebra. The interpretation of the twisted Poincaré algebra as describing relativistic symmetries with clustered two-particle states is proposed.

1. Introduction

Let us consider the Poincaré algebra $\mathcal{P}_4$ with the generators $\hat{g} = (P_\mu, M_{\mu\nu})$ as a classical Hopf algebra. We supplement the well known algebraic relations

$$
\begin{aligned}
&[M_{\mu\nu}, M_{\rho\tau}] = \mathrm{i}(g_{\mu\tau} M_{\nu\rho} - g_{\nu\tau} M_{\mu\rho} + g_{\nu\rho} M_{\mu\tau} - g_{\mu\rho} M_{\nu\tau}) \\
&[M_{\mu\nu}, P_\rho] = \mathrm{i}(g_{\nu\rho} P_\mu - g_{\mu\rho} P_\nu) \\
&[P_\mu, P_\nu] = 0
\end{aligned}
\tag{1.1}
$$

by the 'primitive' coproduct relations

$$
\begin{aligned}
&\Delta_0(M_{\mu\nu}) = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} \\
&\Delta_0(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu
\end{aligned}
\tag{1.2}
$$

and the antipode $S_0(\hat{g}) = -\hat{g}$ ($\hat{g} \in \mathcal{P}_4$). Relations (1.1) lead to the well known Wigner theory of representations of the Poincaré algebra [1, 2] which are spanned by the Hilbert vectors $|m, s; p_\mu, S_3\rangle$, where m and s describe, respectively, the eigenvalues of mass and the relativistic spin (Pauli–Lubanski) Casimir, p_μ is the four-momentum and S_3 ($-S \leqslant S_3 \leqslant S$)

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describes the spin-projection values. The coproduct formula dictates how to calculate the action of the Poincaré generators on the tensor product.

The quantum deformations of the Poincaré algebra are described by modifications to the relations (1.1), (1.2) preserving the Hopf-algebra structure (for the general framework see, e.g., [3,4]). In this paper we would like to consider the mildest quantum deformations of (1.1), (1.2) obtained by the twisting procedure [5–9]. Following Drinfeld [5], two Hopf algebras $\mathcal{A} = (A, \Delta, S, \varepsilon)$ and $\mathcal{A}^F = (A, \Delta^F, S^F, \varepsilon)$ are related by twisting if there exists an invertible function $F = \sum_i f_i \otimes f^i \in \mathcal{A} \otimes \mathcal{A}$† satisfying the 'cocycle' condition [5,7,8]

$$F_{23}(1 \otimes \Delta)F = F_{12}(\Delta \otimes 1)F \tag{1.3}$$

and $(\varepsilon \otimes 1)F = (1 \otimes \varepsilon)F = 1$. In such a case Δ^F and Δ are related as follows ($a \otimes b \cdot c \otimes d = ac \otimes bd$)

$$\Delta^F(a) = F \cdot \Delta(a) \cdot F^{-1}. \tag{1.4a}$$

Introducing $U = \sum_i f_i \cdot S(f^i)$ one also obtains that

$$S^F(a) = US(a)U^{-1}. \tag{1.4b}$$

If $\mathcal{A}$ is the quasi-triangular Hopf algebra and relations (1.3) are replaced by [5,6]

$$(\Delta \otimes 1)F = F_{13}F_{23} \qquad (1 \otimes \Delta)F = F_{13}F_{12} \tag{1.5}$$

then the universal R-matrices for $\mathcal{A}$ and $\mathcal{A}^F$ are related by the formulae ($\tilde{F} = \sigma \cdot F = \sum_i f^i \otimes f_i$)

$$R^F = F^{-1} \cdot R \cdot \tilde{F}. \tag{1.6}$$

For the complex simple Lie algebras $\hat{g}$, twistings described by

$$F = \exp f \qquad f \in \hat{c} \otimes \hat{c} \tag{1.7}$$

where $\hat{c}$ is a subalgebra of $\hat{g}$ (Cartan subalgebra in [6], Borel subalgebra in [8]) were considered. It is easy to check that if $f \in \hat{c} \otimes \hat{c}$ and $\hat{c}$ is Abelian, then conditions (1.5) are valid.

In this paper we shall consider the twisting of the physically important case of inhomogeneous rotation algebras $\hat{g} = O(D-k, k) \ni\!\!\!\supset T_D$, in particular the $D = 4$ Poincaré algebra $\hat{g} = O(3,1) \ni\!\!\!\supset T_4$. In such non-simple algebras one can select a commutative subalgebra $\hat{c}$ in several ways, using, e.g.,

(i) a Cartan subalgebra $(h_1, \ldots, h_n)$ ($n = \frac{1}{2}D$ for D even, $n = \frac{1}{2}(D-1)$ for D odd);

(ii) translation generators $(P_1 \ldots P_D)$; or

(iii) a 'mixed' Cartan–translation algebra C_k ($k \leqslant \frac{1}{2}D$)

$$C_k = (h_1 \ldots h_k, P_{2k+1} \ldots P_D). \tag{1.8}$$

The aim of this paper is (i) to describe the twistings of $U_q(\mathcal{P}_4)$ depending on Cartan generators and translation generators and (ii) to provide an interesting example.

In section 2 we shall consider in an explicit way the twisted $D = 4$ Poincaré algebras $U^F(\mathcal{P}_4)$ with the choice of algebra $\hat{g}$ (see (1.7)) described by formula (1.8) with $k = 0, 1, 2$. Further generalization in the presence of central generators Z_i ($[Z_i, \hat{g}] = 0$ for $\hat{g} \in \mathcal{P}_4$) is also given. In section 3 we shall discuss, as an example of a classical twisted Poincaré algebra, the quantum Poincaré algebra considered recently by Chaichian and Demiczev [10]. In section 4 we shall discuss the elements of the representation theory of twisted Poincaré algebras and present an outlook: some generalizations as well as unsolved problems.

† Strictly speaking, we consider below F as belonging to an extension of $\mathcal{A} \otimes \mathcal{A}$.

2. Twisting of the classical Poincaré algebra

Let us denote the basis of the commutative algebra $\hat{c}(F \in \hat{c} \otimes \hat{c})$ by $(c_1 \ldots c_n)$. We define

$$F = F_+ F_- \qquad F_\pm = \exp f_\pm \tag{2.1}$$

where $f_\pm = \pm\sigma \cdot f_\pm$ (σ is the exchange map: $\sigma(c_i \otimes c_j) = c_j \otimes c_i$) and

$$f^{(\pm)} = \tfrac{1}{2}\alpha_{ij}^{(\pm)}(c_i \otimes c_j \pm c_j \otimes c_i) \tag{2.2}$$

i.e. one can assume that $\alpha_{\pm ij} = \pm\alpha_{\pm ji}$.

If we twist the coproducts of the classical Lie algebra we obtain from the commutativity of $\hat{c}$ that

$$U = \sum f_i \cdot S(f^i) = \exp(-\alpha_{+ij} c_i c_j) \tag{2.3}$$

and after using (1.6) the R-matrix takes the particular form

$$R = \exp(-2f_-) = (F_-)^{-2}. \tag{2.4}$$

The formulae for the coproduct Δ^F depend on the particular choice of the algebra $\hat{c}$. We shall further specify our algebra for the case of classical Poincaré algebra (1.1) and we shall consider the following three types of twist function:

(i) $\hat{c} = (M_3 = M_{12}, N_3 = M_{30})$.
We postulate

$$\begin{aligned} f_+ &= \alpha_+ M_3 \otimes M_3 + \beta_+(M_3 \otimes N_3 + N_3 \otimes M_3) + \gamma_+ N_3 \otimes N_3 \\ f_- &= \beta_-(M_3 \otimes N_3 - N_3 \otimes M_3). \end{aligned} \tag{2.5}$$

One obtains ($M_i \equiv \frac{1}{2}\epsilon_{ijk}M_{jk}$; $M_\pm \equiv M_1 \pm \mathrm{i}M_2$; $N_i \equiv M_{i0}$; $P_\pm \equiv P_1 \pm \mathrm{i}P_2$)

$$\begin{aligned} \Delta^F(M_\pm) &= M_\pm \otimes \mathrm{e}^{\pm A_1}\cos(B_1) + \mathrm{e}^{\pm A_2}\cos(B_2) \otimes M_\pm \\ &\qquad \pm N_\pm \otimes \mathrm{e}^{\pm A_1}\sin(B_1) \pm \mathrm{e}^{\pm A_2}\sin(B_2) \otimes N_\pm \\ \Delta^F(M_3) &= M_3 \otimes 1 + 1 \otimes M_3 \\ \Delta^F(N_\pm) &= N_\pm \otimes \mathrm{e}^{\pm A_1}\cos(B_1) + \mathrm{e}^{\pm A_2}\cos(B_2) \otimes N_\pm \\ &\qquad \mp M_\pm \otimes \mathrm{e}^{\pm A_1}\sin(B_1) \mp \mathrm{e}^{\pm A_2}\sin(B_2) \otimes M_\pm \\ \Delta^F(N_3) &= N_3 \otimes 1 + 1 \otimes M_3 \\ \Delta^F(P_\pm) &= P_\pm \otimes \mathrm{e}^{\pm A_1} + \mathrm{e}^{\pm A_2} \otimes P_\pm \\ \Delta^F(P_3) &= P_3 \otimes \cos(B_1) + \cos(B_2) \otimes P_3 + \mathrm{i}P_0 \otimes \sin(B_1) + \mathrm{i}\sin(B_2) \otimes P_0 \\ \Delta^F(P_0) &= P_0 \otimes \cos(B_1) + \cos(B_2) \otimes P_0 + \mathrm{i}P_3 \otimes \sin(B_1) + \mathrm{i}\sin(B_2) \otimes P_3 \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} A_k &= \alpha_+ M_3 + (\beta_+ - (-1)^k \beta_-) N_3 \\ B_k &= \gamma_+ N_3 + (\beta_+ + (-1)^k \beta_-) M_3. \end{aligned}$$

(ii) $\hat{c} = (M_3 = M_{12}, P_3, P_0)$.

We assume that $(r, s = 3, 0)$

$$
\begin{aligned}
f_+ &= \alpha_+ M_3 \otimes M_3 + \delta_+^r (M_3 \otimes P_r + P_r \otimes M_3) + \rho_+^{rs} P_r \otimes P_s \\
f_- &= \delta_-^r (M_3 \otimes P_r - P_r \otimes M_3).
\end{aligned}
\tag{2.7}
$$

One obtains

$$
\begin{aligned}
\Delta^F(M_\pm) &= M_\pm \otimes \mathrm{e}^{\pm A_1} + \mathrm{e}^{\pm A_2} \otimes M_\pm \pm P_\pm \otimes B_3 \mathrm{e}^{\pm A_1} \pm \mathrm{e}^{\pm A_2} C_3 \otimes P_\pm \\
\Delta^F(M_3) &= M_3 \otimes 1 + 1 \otimes M_3 \\
\Delta^F(N_\pm) &= N_\pm \otimes \mathrm{e}^{\pm A_1} + \mathrm{e}^{\pm A_2} \otimes N_\pm - \mathrm{i} P_\pm \otimes B_0 \mathrm{e}^{\pm A_1} + C_0 \mathrm{e}^{\pm A_2} \otimes P_\pm \\
\Delta^F(N_3) &= N_3 \otimes 1 + 1 \otimes N_3 - \mathrm{i} P_3 \otimes B_0 + C_0 \otimes P_3 + P_0 \otimes B_3 + C_3 \otimes P_0 \\
\Delta^F(P_1) &= P_1 \otimes \cosh(A_1) + \cosh(A_2) \otimes P_1 + \mathrm{i} P_2 \otimes \sinh(A_1) + \mathrm{i} \sinh(A_2) \otimes P_2 \\
\Delta^F(P_2) &= P_2 \otimes \cosh(A_1) + \cosh(A_2) \otimes P_2 - \mathrm{i} P_1 \otimes \sinh(A_1) - \mathrm{i} \sinh(A_2) \otimes P_1 \\
\Delta^F(P_3) &= P_3 \otimes 1 + 1 \otimes P_3 \\
\Delta^F(P_0) &= P_0 \otimes 1 + 1 \otimes P_0
\end{aligned}
\tag{2.8}
$$

where

$$
\begin{aligned}
A_1 &= \alpha_+ M_3 + (\delta_+^r + \delta_-^r) P_r \qquad & B_r &= (\delta_+^r - \delta_-^r) M_3 + \rho_+^{rs} P_s \\
A_2 &= \alpha_+ M_3 + (\delta_+^r - \delta_-^r) P_r \qquad & C_r &= \rho_+^{sr} P_s + (\delta_+^r + \delta_-^r) M_3.
\end{aligned}
$$

(iii) $\hat{c} = (P_1, P_2, N_3 = M_{30})$.

Putting $(a, b = 1, 2)$

$$
\begin{aligned}
f_+ &= \rho_+^{ab} P_a \otimes P_a + \xi_+^a (N_3 \otimes P_a + P_a \otimes N_3) + \gamma_+ N_3 \otimes N_3 \\
f_- &= \xi_-^a (N_3 \otimes P_a - P_a \otimes N_3).
\end{aligned}
\tag{2.9}
$$

One obtains

$$
\begin{aligned}
\Delta^F(M_\pm) &= M_\pm \otimes \cos(A_1) + \cos(A_2) \otimes M_\pm \pm \{N_\pm \otimes \sin(A_1) + \sin(A_2) \otimes N_\pm\} \\
&\quad \mp \{P_3 \otimes (B_1 \pm \mathrm{i} B_2) \cos(A_1) + (C_1 \pm \mathrm{i} C_2) \cos(A_2) \otimes P_3\} \\
&\quad \mp \mathrm{i}\{P_0 \otimes (B_1 \pm \mathrm{i} B_2) \sin(A_1) + (C_1 \pm \mathrm{i} C_2) \sin(A_2) \otimes P_0\} \\
\Delta^F(M_3) &= M_3 \otimes 1 + 1 \otimes M_3 \\
&\quad - \mathrm{i}\{P_2 \otimes B_1 + C_1 \otimes P_2\} + \mathrm{i}\{P_1 \otimes B_2 + C_2 \otimes P_1\} \\
\Delta^F(N_\pm) &= N_\pm \otimes \cos(A_1) + \cos(A_2) \otimes N_\pm \mp \{M_\pm \otimes \sin(A_1) + \sin(A_2) \otimes M_\pm\} \\
&\quad - \mathrm{i}\{P_0 \otimes (B_1 \pm \mathrm{i} B_2) \cos(A_1) + (C_1 \pm \mathrm{i} C_2) \cos(A_2) \otimes P_0\} \\
&\quad + P_3 \otimes (B_1 \pm \mathrm{i} B_2) \sin(A_1) + (C_1 \pm \mathrm{i} C_2) \sin(A_2) \otimes P_3 \\
\Delta^F(N_3) &= N_3 \otimes 1 + 1 \otimes N_3 \\
\Delta^F(P_\pm) &= P_\pm \otimes 1 + 1 \otimes P_\pm \\
\Delta^F(P_3) &= P_3 \otimes \cos(A_1) + \cos(A_2) \otimes P_3 + \mathrm{i} P_0 \otimes \sin(A_1) + \mathrm{i} \sin(A_2) \otimes P_0 \\
\Delta^F(P_0) &= P_0 \otimes \cos(A_1) + \cos(A_2) \otimes P_0 + \mathrm{i} P_3 \otimes \sin(A_1) + \mathrm{i} \sin(A_2) \otimes P_3
\end{aligned}
\tag{2.10}
$$

where

$$B_1 \pm iB_2 = (\rho_+^{1b} \pm i\rho_+^{2b})P_b + (\xi_+^1 \pm i\xi_+^2 - (\xi_-^1 \pm i\xi_-^2))N_3$$

$$C_1 \pm iC_2 = (\rho_+^{1b} \pm i\rho_+^{2b})P_b + (\xi_+^1 \pm i\xi_+^2 + (\xi_-^1 \pm i\xi_-^2))N_3.$$

(iv) $\hat{c} = (P_1, P_2, P_3, P_0)$.

One can write $(\rho_\pm^{\mu\nu} = \pm\rho_\pm^{\nu\mu})$

$$\begin{aligned} f_+ &= \rho_+^{\mu\nu}(P_\mu \otimes P_\nu + P_\nu \otimes P_\mu) \\ f_- &= \rho_-^{\mu\nu}(P_\mu \otimes P_\nu - P_\nu \otimes P_\mu). \end{aligned} \tag{2.11}$$

Because the split Casimir

$$C_2^{\text{split}} \equiv \Delta(P_\mu P^\mu) - P_\mu P^\mu \otimes 1 - 1 \otimes P_\mu P^\mu = 2P_\mu \otimes P^\mu \tag{2.12}$$

commutes with $\Delta(\hat{a})$ for any $\hat{a} \in U(\mathcal{P}_4)$, one can assume further that $\rho_+^{\mu\nu}\eta_{\mu\nu} = \rho_\mu^\mu = 0$ $(\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1))$.

The formulae for the coproduct take the form

$$\begin{aligned} \Delta^F(M_{\mu\nu}) &= M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} \\ &\quad + (\alpha_{+\mu}{}^\rho P_\nu - \alpha_{+\nu}{}^\rho P_\mu) \otimes P_\rho + P_\rho \otimes (\alpha_{+\mu}{}^\rho P_\nu - \alpha_{+\nu}{}^\rho P_\mu) \\ &\quad + (\alpha_{-\mu}{}^\rho P_\nu - \alpha_{-\nu}{}^\rho P_\mu) \otimes P_\rho - P_\rho \otimes (\alpha_{-\mu}{}^\rho P_\nu - \alpha_{-\nu}{}^\rho P_\mu) \\ \Delta^F(P_\mu) &= P_\mu \otimes 1 + 1 \otimes P_\mu. \end{aligned} \tag{2.13}$$

In general we assume that the Poincaré algebra is the complex one and that the twist function parameters are also complex. The reality condition imposed on the Poincaré generators imply the reality conditions for the coefficients in formulae (2.5), (2.7), (2.9) and (2.11). For simplicity we shall consider the last example of the twist function, given by (2.11). It is known that if the real structure is an antihomomorphism in the algebra sector, one can still impose two types of reality conditions on the generators of twisted Poincaré algebra [10, 11]:

(i) The standard one, denoted in [11] by +. In the case of formulae (2.13) one obtains

$$(\Delta(M_{\mu\nu}))^+ = \Delta(M_{\mu\nu}) \Longrightarrow \alpha^{\rho\tau} \text{ real.} \tag{2.14a}$$

(ii) The non-standard one, used, e.g., in [12], and denoted in [11] by $\oplus$. In such a case

$$(\Delta(M_{\mu\nu}))^\oplus = \Delta(M_{\mu\nu}) \Longrightarrow \alpha^{\rho\tau} = (\alpha^{\tau\rho})^* \tag{2.14b}$$

i.e. the matrix $\alpha \equiv (\alpha^{\rho\tau})$ is Hermitian.

Finally we consider the extension of $\hat{g}$ by an Abelian algebra $\hat{z}(\hat{g} \to \hat{g} \oplus \hat{z})$ with $z_A(A = 1, \ldots, m)$ describing the central charges. The formulae (2.2) determining the twist function can be extended as follows

$$f_\pm \to f_\pm^{(z)} = f_\pm + \tfrac{1}{2}\beta_{\pm iA}(c_i \otimes Z_A \pm Z_A \otimes c_i). \tag{2.15}$$

The candidates for Z_A are the central charges as well as the Casimir operators. As an example, we shall consider case (iv) with one central charge Z, i.e. we assume that formulae (2.11) are extended as follows

$$f_\pm \to f_\pm^{(z)} = f_\pm + \rho_\pm^\mu (P_\mu \otimes Z \pm Z \otimes P_\mu). \tag{2.16}$$

The formulae (2.13) for a twisted coproduct are modified as follows

$$\Delta^F(M_{\mu\nu}) \to \Delta^F(M_{\mu\nu}) + \rho_+^\mu (P_\mu \otimes Z + Z \otimes P_\mu) + \rho_-^\mu (P_\mu \otimes Z - Z \otimes P_\mu). \tag{2.17}$$

With the choice (2.16), the explicit formula for the universal R-matrix is given by

$$\begin{aligned} R &= \exp(-2 f_-^{(z)}) \\ &= \exp(-2\rho_-^{\mu\nu}(P_\mu \otimes P_\nu P_\nu \otimes P_\mu)) \\ &= \exp(-2\rho_-^\mu (P_\mu \otimes Z - Z \otimes P_\mu)). \end{aligned} \tag{2.18}$$

The invariant tensor (2.3) takes the form

$$U = \exp(-2\alpha_+^{\mu\nu} P_\mu \cdot P_\nu - 2\rho_+^\mu P_\mu \cdot Z) \tag{2.19}$$

and using the formula $S^F = U S_0 U^{-1}$ one obtains

$$\begin{aligned} S^{F^{(z)}}(P_\mu) &= S_0(P_\mu) = -P_\mu \\ S^{F^{(z)}}(M_{\mu\nu}) &= -M_{\mu\nu} - 2(\alpha_{+\mu}{}^\rho P_\rho - \alpha_\nu{}^\rho P_\mu P_\rho) - (\rho_{+\mu} P_\nu - \rho_{+\nu} P_\mu) \cdot Z. \end{aligned} \tag{2.20}$$

The reality conditions for the parameters $\rho_\pm^\mu$ take the form

$$\begin{aligned} &\text{(i)} \qquad + - \text{involution} : \rho_\pm^\mu \text{ real} \\ &\text{(ii)} \qquad \oplus - \text{involution} : (\rho_+^\mu)^* = \rho_-^\mu. \end{aligned} \tag{2.21}$$

In this section we consider classical twisted Poincaré algebras, parametrized by multiparameter twist functions. These Hopf algebras use duality relations to determine multiparameter deformations of the functions of the Poincaré group. Using the duality relation between multiplication and comultiplication

$$\langle a \cdot b, c \rangle = \langle a \otimes b, \Delta(c) \rangle \tag{2.22}$$

one sees easily that all the antisymmetric contributions to the twisted coproducts (see equation (2.13)) lead to non-commutativity of the generators of the corresponding dual quantum Poincaré group.

It is an interesting exercise to compare the dual quantum Poincaré groups with the classical twisted Poincaré algebras.

3. An example: Chaichian–Demiczev quantum Poincaré algebra

We shall show that the example of a q-Poincaré algebra, given in [10], is isomorphic as a Hopf algebra to the twisted classical Poincaré algebra. First, we shall describe the complex classical Lorentz algebra $SO(4;C) = SO(3;C) \oplus SO(3;C)$ as follows

$$[e_i, e_{-j}] = \delta_{ij} h_i \qquad [h_i, h_j] = 0 \qquad [h_i, e_{\pm j}] = \pm 2\delta_{ij} e_{\pm j} \tag{3.1}$$

where (e_1, e_{-1}, h_1) and (e_2, e_{-2}, h_2) describe two $O(3;C)$ sectors.

Introducing

$$\begin{aligned} L_1 &= e_{-1} \qquad & L_2 &= e_{-2} \qquad & L_5 &= \tfrac{1}{2}(h_1 + h_2) \\ L_3 &= e_{+2} \qquad & L_4 &= e_{+1} \qquad & L_6 &= \tfrac{1}{2}(h_2 - h_1) \end{aligned} \tag{3.2}$$

one obtains the relations

$$\begin{aligned} &[L_1, L_5] = L_1 \qquad & &[L_2, L_5] = L_2 \\ &[L_1, L_6] = -L_1 \qquad & &[L_2, L_6] = L_2 \\ &[L_1, L_4] = L_6 - L_5 \qquad & &[L_2, L_3] = L_6 + L_5 \\ &[L_3, L_5] = -L_3 \qquad & &[L_4, L_5] = -L_4 \\ &[L_3, L_6] = -L_3 \qquad & &[L_4, L_6] = L_4 \\ &[L_1, L_2] = [L_1, L_3] = [L_2, L_4] = [L_3, L_4] = [L_5, L_6] = 0 \end{aligned} \tag{3.3}$$

where $(a = 1, \ldots, 6)$

$$\Delta(L_a) = L_a \otimes I + I \otimes L_a. \tag{3.4}$$

Let us perform the twist of this coproduct

$$F = q^{h_2 \otimes h_1} = q^{(L_5 + L_6)\otimes(L_5 - L_6)}.$$

One gets $(\Delta^F(L_a) = F \cdot \Delta(L_a) \cdot F^{-1})$

$$\begin{aligned} \Delta^F(L_1) &= L_1 \otimes I + q^{-2(L_5+L_6)} \otimes L_1 \\ \Delta^F(L_2) &= I \otimes L_2 + L_2 \otimes q^{-2(L_5-L_6)} \\ \Delta^F(L_3) &= I \otimes L_3 + L_3 \otimes q^{2(L_5-L_6)} \\ \Delta^F(L_4) &= L_4 \otimes I + q^{2(L_5+L_6)} \otimes L_4 \\ \Delta^F(L_5) &= \Delta(L_5) \\ \Delta^F(L_6) &= \Delta(L_6). \end{aligned} \tag{3.5}$$

Introducing

$$\begin{aligned} &\tilde{L}_1 = L_1 \qquad \tilde{L}_2 = q^{-2} L_2 q^{-2(L_5-L_6)} \\ &\tilde{L}_3 = q^{-2} L_3 q^{2(L_5-L_6)} \qquad \tilde{L}_4 = L_4 \\ &\tilde{L}_5 = L_5 \qquad \tilde{L}_6 = L_6 \end{aligned} \tag{3.6}$$

one can identify the transformed classical Lorentz algebra (3.3) with the q-deformed Lorentz algebra proposed in [10] with the coproduct

$$\begin{aligned}
\Delta^F(\tilde{L}_1) &= \tilde{L}_1 \otimes I + q^{-2(\tilde{L}_5+L_6)} \otimes \tilde{L}_1 \\
\Delta^F(\tilde{L}_2) &= I \otimes \tilde{L}_2 + \tilde{L}_2 \otimes q^{-2(\tilde{L}_5-\tilde{L}_6)} \\
\Delta^F(\tilde{L}_3) &= I \otimes \tilde{L}_3 + \tilde{L}_3 \otimes q^{2(\tilde{L}_5-\tilde{L}_6)} \\
\Delta^F(\tilde{L}_4) &= \tilde{L}_4 \otimes I + q^{2(\tilde{L}_5+\tilde{L}_6)} \otimes \tilde{L}_4 \\
\Delta^F(\tilde{L}_5) &= \tilde{L}_5 \otimes I + I \otimes \tilde{L}_5 \\
\Delta^F(\tilde{L}_6) &= \tilde{L}_6 \otimes I + I \otimes \tilde{L}_6.
\end{aligned} \tag{3.7}$$

Introducing four-momentum operators, which in the basis (3.2) will satisfy the following covariance relations with L_5, L_6,

$$\begin{aligned}
&[P_1, L_5] = P_1 \qquad [P_2, L_5] = 0 \\
&[P_3, L_5] = -P_3 \qquad [P_4, L_5] = 0 \\
&[P_1, L_6] = 0 \qquad [P_2, L_6] = P_2 \\
&[P_3, L_6] = 0 \qquad [P_4, L_6] = -P_4
\end{aligned} \tag{3.8}$$

one obtains after the nonlinear transformation

$$\begin{aligned}
&\tilde{P}_1 = q^{L_5-L_6} P_1 \qquad \tilde{P}_2 = q^{L_5-L_6} P_2 \\
&\tilde{P}_3 = q^{L_6-L_5} P_3 \qquad \tilde{P}_4 = q^{L_6-L_5} P_4
\end{aligned} \tag{3.9}$$

the relations

$$\begin{aligned}
&[\tilde{P}_1, \tilde{P}_2]_{q^2} = [\tilde{P}_4, \tilde{P}_1]_{q^2} = [\tilde{P}_2, \tilde{P}_3]_{q^2} = [\tilde{P}_3, \tilde{P}_4]_{q^2} = 0 \\
&[\tilde{P}_1, \tilde{P}_3] = [\tilde{P}_2, \tilde{P}_4] = 0
\end{aligned} \tag{3.10}$$

and the coproducts

$$\begin{aligned}
\Delta^F(\tilde{P}_1) &= \tilde{P}_1 \otimes 1 + q^{-2L_6} \otimes \tilde{P}_1 \\
\Delta^F(\tilde{P}_2) &= \tilde{P}_2 \otimes 1 + q^{2L_5} \otimes \tilde{P}_2 \\
\Delta^F(\tilde{P}_3) &= \tilde{P}_3 \otimes 1 + q^{2L_6} \otimes \tilde{P}_3 \\
\Delta^F(\tilde{P}_4) &= \tilde{P}_4 \otimes 1 + q^{-2L_5} \otimes \tilde{P}_4.
\end{aligned} \tag{3.11}$$

The relations (3.10), (3.11) describe the translation sector of the Chaichian–Demiczev quantum algebra.

Let us recall that recently the quantum Lorentz groups have been classified by Wororonowicz and Zakrzewski [13], where, besides the Drinfeld–Jimbo parameter q, a new parameter t was introduced. It can be shown that the quantum deformation, proposed by Chaichian and Demiczev, corresponds to $q = 1$. This condition, as the necessary requirement for the existence of a non-trivial quantum deformation of the Poincaré algebra, with the Lorentz part as the Hopf subalgebra, has been obtained in [14] (see also [15]).

It should be stressed that in [13] there were also given other examples of the quantum deformations of the Lorentz group which satisfied the condition $q = 1$ and could be extended to the quantum deformations of the Poincaré algebra without supplementing an eleventh dilatation generator. It would be interesting to prove the conjecture that all quantum deformations of the Poincaré algebra which have the deformed Lorentz algebra as their Hopf subalgebra are classical twisted Poincaré algebras.

We would like finally to mention that it is possible to obtain the Poincaré quantum group as well as the Poincaré quantum algebra with the Drinfeld–Jimbo deformation parameter $q \neq 1$, if we assume a braided structure for the tensor products, i.e. we consider the deformations in the framework of braided quantum groups and algebras (see, e.g., [16]). In such a case the parameter q enters into the definition of the braided tensor product of the Lorentz generators and the translation generators [14] (see also [17, 18]). In this paper, however, we assume the standard 'bosonic' relations for the tensor categories.

4. Discussion

4.1. Representation theory of twisted Poincaré algebra

The theory of irreducible representations of twisted Poincaré algebras is described by the conventional Wigner representations for the Poincaré algebra [1, 2]. The twisting can be interpreted as a modification of the tensor products for relativistic free particle states, in particular the two-particle sectors in a relativistic Fock space. The tensor product $|1\rangle \otimes |2\rangle$ of two free one-particle states ($i = 1, 2$)

$$|i\rangle = |m^{(i)}, s^{(i)}; p_\mu^{(i)}, s_3^{(i)}\rangle \tag{4.1}$$

can be modified as follows

$$|1\rangle \otimes_F |2\rangle = F(c^{(1)}, c^{(2)})|1\rangle \otimes |2\rangle \tag{4.2a}$$

where ($\alpha = \alpha_+ + \alpha_-$)

$$F(c^{(1)}, c^{(2)}) = \exp \alpha_{ij} c_i^{(1)} c_j^{(2)}. \tag{4.2b}$$

Let $\hat{\alpha}$ denote the algebra describing the levels of the representation space (for (4.1) $\hat{\alpha} = (P_\mu, S_3)$, where $S_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\tau} M^{\nu\rho} P^\tau$) and let $\hat{O}$ denote the Casimirs parametrizing by their eigenvalues the representations $\hat{O} = (P_\mu P^\mu, S_\mu S^\mu)$ for $\mathcal{P}_4$. One can distinguish the following two cases:

(i) $[c_i, \hat{\alpha}] = 0$. This corresponds to our choice (iv) (see (2.11), (2.13)). In such a case the twisted tensor product of two representations (4.1) describes the fixed four-momenta components of the wavepacket

$$|1, 2\rangle_F = \exp(\alpha^{\mu\nu} p_\mu^{(1)} p_\nu^{(2)})|1\rangle \otimes |2\rangle. \tag{4.3}$$

For dimensional reasons one should put $\alpha^{\mu\nu} = (1/\kappa^2) a^{\mu\nu}$ (κ-mass-like parameter). If we assume that $a^{\mu\nu}$ has negative eigenvalues, one obtains from (4.3) the Gauss-like two-particle wavefunction.

(ii) $[c_i, \hat{\alpha}] \neq 0$. Such a case is decribed by the choices (i), (ii), (iii) of the twist function given in section 2 as well as the example described in section 3. In such a case, twisted two-particle states described by (4.2) are not eigenvalues of the 'two-particle observable' $\Delta^F(\hat{\alpha})$, because

$$\Delta^F(\hat{\alpha}) = F \cdot \Delta(\hat{\alpha}) \cdot F^{-1} \neq \Delta(\hat{\alpha}). \tag{4.4}$$

For the four-momentum operators the additivity of the four-momenta eigenvalues is modified by the formula

$$\Delta^F(P_\mu) = F \cdot (P_\mu \otimes 1 + 1 \otimes P_\mu) \cdot F^{-1}. \tag{4.5}$$

In our example in section 3, formula (4.5) takes the form (3.11). The physical interpretation of generalized wavepackets (4.2a) with the modified addition for the four-momenta is not clear.

4.2. *Twisted Poincaré algebra from the contraction of $U_q(O(4,2))$*

In a recent paper [12] two of the present authors proposed the contraction of $U_q(O(4,2))$ to a quantum Poincaré algebra. It can be shown that the result of the contraction is a twisted Poincaré algebra with the twist function depending on the four-momenta and one central charge Z (see (2.16)) obtained from the contraction of the dilatation generator in the conformal algebra.

4.3. *Non-Abelian choice of twist functions*

It is interesting to consider more general classes of twisting functions with F spanned by non-Abelian sectors of the algebra. In particular, such a twisting function is provided by the universal R-matrix, which interchanges two non-cocomutative coproducts Δ and $\Delta' = \sigma \cdot \Delta$ of a quantum algebra. It is known that for Drinfeld–Jimbo deformations $U_q(\hat{g})$ of simple Lie algebras the universal R-matrix can be decomposed into the product [19, 20]

$$R = \prod_{\alpha \in \Delta^{(+)}} R_\alpha \cdot K \tag{4.6}$$

where

$$R_\alpha = \exp q_\alpha (a_\alpha(q) e_\alpha \otimes e_{-\alpha}) \tag{4.7}$$

and K depends only on the Cartan generators. It appears that any component (4.7) of the product (4.6) can be used as a twist function F [21]. Because $a_\alpha(q)$ is proportional to $q - q^{-1}$, the twisting with $F = R_\alpha$ can be introduced only for genuine quantum algebras ($q \neq 1$). It is interesting to find non-trivial twist functions for quantum κ-Poincaré algebra proposed in [22, 23]. Because the universal $\hat{R}$-matrix for κ-Poincaré algebra is not known, the type of twisting proposed in [21] cannot be applied.

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